Rate of Convergence Estimates for Random Polarizations on $\mathbb{R}^d$

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Abstract

It is shown in [1] that the expected L^1 distance between f^* and n random polarizations of a function f with support in a ball of radius L is bounded above by $2dm(B_{2L})\|f\|_\infty n^{-1}$. Here it is shown that the expected distance is bounded below by $b_d^n\|f - f^*\|_{L^1}$ with $b_d > 2^{-1}$ some numerical constant depending only on d and, if the boundary of the sets $\{f > t\}$ have measure zero for almost every t, equals $n^{-1}c_n$ with $\limsup_{n\to\infty} c_n \le L2^{d+1}\|\mathrm{Per}(f^* > t)\|_{L^1}$. The paper concludes with the study of random polarizations of balls which shows that the aformentioned $O(n^{-1})$ rate of convergence estimate is sharp for $d = 1$.

1 Introduction

It is well known that for any function $f \in L^p$ $(1 \le p < \infty)$ there exists a sequence of polarizations that, when applied iteratively to f, will yield a sequence of functions which converges in L^p to f^* - the symmetric decreasing rearrangement of f. In fact, one can construct explicit sequences of polarizations that yield convergence to the symmetric decreasing rearrangement for any initial function in L^p [3]. These sequences also yield uniform convergence to the symmetric decreasing rearrangement when applied to any initial continuous function with compact support and also convergence in Hausdorff metric when applied to compact sets. Other convergence results have been proven. For instance, it is shown in [1] and [2] that random sequences of polarizations can also yield almost sure convergence to the symmetric decreasing rearrangement. The first result on rates of convergence of polarizations to the symmetric decreasing rearrangement appears in [1]. It is shown in

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that paper that there exists a probability measure $\mathbb{P}_L$ defined on the space of infinite sequences of polarizations with the following rate of convergence property:

Proposition 1. *[1, p.19]If $f \in L^1(\mathbb{R}^d)$ is bounded with support in B_L then*

$$\mathbb{E}_{\mathbb{P}_L}\left(\|f^{\sigma_1\cdots\sigma_n} - f^*\|_{L^1}\right) \leq 2dm(B_{2L})\|f\|_\infty n^{-1}. \tag{1}$$

It is shown in this paper that the following holds:

Proposition 2. *If $f \in L^1$ with support in B_L then*

$$b_d^n\|f - f^*\|_{L^1} \leq \mathbb{E}_{\mathbb{P}_L}\left(\|f^{\sigma_1\cdots\sigma_n} - f^*\|_{L^1}\right) = n^{-1}c_n \tag{2}$$

with $b_d > 2^{-1}$ a numerical constant that depends only on d and

$$\limsup_{n\to\infty} c_n \leq L2^{d+1}\int_0^\infty \mathrm{Per}\left(\overline{\{f > t\}}^*\right) dt. \tag{3}$$

2 Notation and Preliminary Results

2.1 Notation

In what follows, m will denote Lebesgue measure; $\mathcal{M}$ will denote the sigma algebra of Lebesgue measurable sets; B_t will denote the ball of radius t centered at the origin and ω_d will denote the surface area of B_1. The space of reflections that do not map the origin to the origin will be denoted by Ω. $\sigma_{x,y}$ will denote the unique reflection that maps x to y. Finally, $R(A)$ will denote the outer radius of a set A and $d_H(\cdot,\cdot)$ will denote the Hausdorff distance between two sets.

2.2 Rearrangements

A rearrangement T is a map $T : \mathcal{M} \to \mathcal{M}$ that is both *monotone* ($A \subset B$ implies $T(A) \subset T(B)$) and *measure preserving* ($m(T(A)) = m(A)$ for all $A \in \mathcal{M}$). We say that a non-negative measurable function f *vanishes at infinity* if

$$m(f > t) < \infty \tag{4}$$

for all $t > 0$. If f vanishes at infinity then we can define its rearrangement Tf by using the "layer cake principle"

$$Tf(x) = \int_0^\infty \chi_{T(f>t)}(x)dt = \sup\{t : x \in T(f > t)\}. \tag{5}$$

2.3 Polarization

Let $\sigma \in \Omega$ and let f be an arbitrary function. We define the *polarization of f with respect to σ* as

$$f^\sigma(x) = \begin{cases} f(x) \vee f(\sigma(x)) & \text{if } x \in X_+^\sigma \\ f(x) \wedge f(\sigma(x)) & \text{if } x \in X_-^\sigma \\ f(x) & \text{if } x \in X_0^\sigma . \end{cases} \tag{6}$$

If A is an arbitrary set then its polarization with respect to σ is simply the polarization of χ_A and is denoted by A^σ:

$$A^\sigma = (\sigma(A \cap X_-^\sigma) \cap A^c) \cup (\sigma(A \cap X_+^\sigma) \cap A) \cup (A \cap X_+^\sigma) \cup (A \cap X_0^\sigma) . \tag{7}$$

In other words, A^σ is the same as A except that the part of A contained in X_-^σ whose reflection does not lie in A is replaced by its reflection in X_+^σ. As a result, polarization is measure preserving. It is clear from equation 6 that if $f \leq g$ then $f^\sigma \leq g^\sigma$ for all $\sigma \in \Omega$ and thus polarization is monotone on $\mathcal{M}$ i.e., polarization is a rearrangement. One can check directly that $\{f^\sigma > t\} = \{f > t\}^\sigma$ for all $\sigma \in \Omega$ and, by equation 5), f^σ is the rearrangement of f with respect to the polarization rearrangement.

2.4 Schwarz Symmetrization

For any $A \in \mathcal{M}$ there exists a unique open ball centered at the origin A^* with the same measure as A called the *Schwarz symmetrization of A*. If $f(x)$ vanishes at infinity then its rearrangement with respect to the Schwarz rearrangement is denoted by $f^*(x)$. It is clear that $f^*(x)$ is radially decreasing i.e., $f^*(x) \leq f^*(y)$ if $|x| \geq |y|$ and $f(x) = f(y)$ if $|x| = |y|$. In the literature, f^* is also called the *symmetric decreasing rearrangement of f*. If f vanishes at infinity, we let

$$r_f(t) = \left(\frac{m(f > t)}{\kappa_d} \right)^{1/d} \tag{8}$$

denote the radius of the open ball $\{f > t\}^*$. The distribution function of a function f vanishing at infinity is always right continuous and thus so is $r_f(t)$. In particular, we have

$$\{f^* > t\} = \{f > t\}^* \tag{9}$$

for all $t \geq 0$ and thus f^* is right continuous.

2.5 Random Polarizations

A probability measure $\mathbb{P}_L$ on $\prod_{i=1}^{\infty} \Omega$ can be constructed by letting

$$\mathbb{P}_L(A_1 \times \cdots \times A_n) = \prod_{i=1}^{n} (2L\omega_d)^{-1} \int_{\{\sigma(0):\sigma \in A_i\}} |x|^{-(d-1)} dx. \tag{10}$$

The following proposition motivates the choice of $\mathbb{P}_L$.

Proposition 3. *If $x \in \mathbb{R}^d$ then $T_x(y) = \sigma_{x,y}(0)$ has Jacobian*

$$\left(\frac{|T_x(y)|}{|x-y|} \right)^{d-1} \tag{11}$$

and

$$\mathbb{P}_L(\sigma_1(x) \in E) = (2L\omega_d)^{-1} \int_E |x-y|^{-(d-1)}\, dy \tag{12}$$

for every $x \in B_L$.

Proof. See [1, p.18] for the proof of equation 11. Equation 12 follows directly from equation 11 and the change of variables formula for integrals. □

3 Rate of Convergence

Proof of proposition 2. It is shown in [1, p.19] that

$$\mathbb{E}_{\mathbb{P}_L} \left(m \left(\{f > t\} \vartriangle \{f^* > t\} \right) \right) - m \left(\{f^{\sigma_1} > t\} \vartriangle \{f^* > t\} \right) \tag{13}$$

equals

$$2\, \mathbb{E}_{\mathbb{P}_L} \left(m(\{f > t \geq f^*\} \cap \{f^*(\sigma_1(x)) > t \geq f(\sigma_1(x))\}) \right) \tag{14}$$

for any function $f \in L^1$ with support in B_L. Applying Fubini's theorem to equation 14; setting $\sigma(x) = y$ and using Proposition 3 shows that 14 equals

$$(Lw_d)^{-1} \int_{A_0(t)} \int_{B_0(t)} |x-y|^{-(d-1)} dx\, dy \tag{15}$$

with

$$A_0(t) = \{f > t, f^* \leq t\}, B_0(t) = \{f^* > t, f \leq t\}.$$

If

$$A_n(t) = \{f^{\sigma_1 \cdots \sigma_n} > t, f^* \leq t\}, B_n(t) = \{f^* > t, f^{\sigma_1 \cdots \sigma_n} \leq t\} \tag{16}$$

and

$$(L\omega_d)^{-1} m(A_n(t))^{-2} \int_{A_n(t)} \int_{B_n(t)} |x-y|^{-(d-1)} dx\, dy = \phi_{f^{\sigma_1 \cdots \sigma_n}}(t) \tag{17}$$

then, by recalling equation 15, one sees that the sequence

$$z_n(t) = \mathbb{E}_{\mathbb{P}_L}\left(m\left(\{f^{\sigma_1\cdots\sigma_n} > t\} \triangle \{f^* > t\}\right)\right) \tag{18}$$

satisfies the relation

$$z_n(t) - z_{n+1}(t) = z_n^2(t)\alpha_n(t) \tag{19}$$

with $\alpha_n(t) = \mathbb{E}_{\mathbb{P}_L}\left(\phi_{f^{\sigma_1\cdots\sigma_n}}(t)m(A_n(t))^2\right)z_n(t)^{-2}$. This implies that $z_n(t) = n^{-1}c_n(t)$ with

$$c_n(t) \quad = \quad \frac{z_0(t) - z_n(t)}{z_0(t)}\left(n^{-1}\sum_{i=1}^{n}\frac{\alpha_{i-1}(t)}{1-\alpha_{i-1}(t)z_{i-1}(t)}\right)^{-1} \tag{20}$$

and

$$z_n = \mathbb{E}_{\mathbb{P}_L}\left(\|f^{\sigma_1\cdots\sigma_n} - f^*\|_{L^1}\right) = n^{-1}\int_0^\infty c_n(t)dt = n^{-1}c_n. \tag{21}$$

Moreover,

$$\limsup_{n\to\infty} c_n \quad \leq \quad \int_0^\infty \limsup_{n\to\infty} c_n(t)dt \tag{22}$$

$$\leq \quad \int_0^\infty \limsup_{n\to\infty}\frac{1-\alpha_n(t)z_n(t)}{\alpha_n(t)}dt \tag{23}$$

and similarly for the lim inf. By Jensen's inequality,

$$\alpha_n(t) \geq 4^{-1}\mathbb{E}_{\mathbb{P}_L}\left(\phi^{-1}_{f^{\sigma_1\cdots\sigma_n}}(t)\right)^{-1} \tag{24}$$

and

$$\mathbb{E}_{\mathbb{P}_L}\left(\phi^{-1}_{f^{\sigma_1\cdots\sigma_n}}(t)\right) \leq 2^{d-1}L\omega_d\mathbb{E}_{\mathbb{P}_L}\left(R(\{f^{\sigma_1\cdots\sigma_n} > t\})^{d-1}\right). \tag{25}$$

As discussed in the introduction, if τ_n is a sequence of polarizations that transforms any initial L^1 function to its symmetric decreasing rearrangement then $d_H(F^{\tau_1\cdots\tau_n}, F^*) \to 0$ as $n \to \infty$ for any compact set F. In particular,

$$\lim_{n\to\infty}\omega_d\mathbb{E}_{\mathbb{P}_L}\left(R(\{f^{\sigma_1\cdots\sigma_n} > t\})^{d-1}\right) = \mathrm{Per}\left(\overline{\{f > t\}}^*\right) \tag{26}$$

and

$$\limsup_{n\to\infty} c_n \quad \leq \quad L2^{d+1}\int_0^\infty \mathrm{Per}\left(\overline{\{f > t\}}^*\right)dt. \tag{27}$$

For the lower bound on the rate of convergence, we first apply the Riesz rearrangement inequality,

$$\int_{A_n(t)}\int_{B_n(t)}|x-y|^{-(d-1)}dx\,dy \quad \leq \quad \int_{A_n(t)^*}\int_{B_n(t)^*}|x-y|^{-(d-1)}dx\,dy \tag{28}$$

and the right-hand side of 28 equals

$$\left(\frac{m(A_n(t))}{m(B_1)}\right)^{(d+1)/d}\int_{B_1}\int_{B_1}|x-y|^{-(d-1)}dxdy. \tag{29}$$

This implies that

$$z_{n-1}-z_n\leq\int_0^\infty e_{n-1}(t)z_{n-1}(t)dt \tag{30}$$

with

$$e_{n-1}(t)=c_d\frac{\mathbb{E}_{\mathbb{P}_L}\left(m(A_{n-1}(t))^{1+1/d}\right)}{\mathbb{E}_{\mathbb{P}_L}\left(m(A_{n-1}(t))\right)} \tag{31}$$

for some constant c_d only depending on d. By induction, one obtains

$$\begin{aligned} z_n &\geq \int_0^\infty\prod_{i=0}^{n-1}(1-e_i(t))z_0(t)\,dt && (32)\\ &\geq \|z_0(t)\|_{L^1}\left(\int_0^\infty\prod_{i=0}^{n-1}(1-e_i(t))^{1/n}\frac{z_0(t)}{\|z_0(t)\|_{L^1}}\,dt\right)^n && (33)\end{aligned}$$

for all $n\geq 1$. To complete the proof, we have

$$\begin{aligned} c_d &= (2L\omega_d)^{-1}m(B_1)^{-(d+1)/d}\int_{B_1}\int_{B_1}|x-y|^{-(d-1)}dxdy && (34)\\ &= (2L)^{-1}m(B_1)^{-1/d}\gamma_d && (35)\end{aligned}$$

and thus $e_i(t)$ is bounded above by $\gamma_d r_f(t)/2L<2^{-1}\gamma_d$. Setting $b_d=1-2^{-1}\gamma_d$ completes the proof.

□

4 Example

Let x_0 denote the center of a ball with $|x_0|<L-r$ and radius r and let X_n denote the distance from the origin of the center of n random polarizations of x_0+B_r. The functions

$$\phi_n(x)=(2L\omega_d)^{-1}\int_{|y|<|x|}|y|^n|x-y|^{-(d-1)}dy \tag{36}$$

have the scaling property $\phi_n(\lambda x)=\lambda^{n+1}\phi_n(x)$. As a result, if u is any unit vector then

$$\mathbb{E}_{\mathbb{P}_L}\left(X_n^k\right)-\mathbb{E}_{\mathbb{P}_L}\left(X_{n-1}^k\right)=-c_k\mathbb{E}_{\mathbb{P}_L}\left(X_{n-1}^{k+1}\right) \tag{37}$$

with

$$c_k = (2L\omega_d)^{-1} \int_{|y|<1} (1 - |y|^k)|u - y|^{-(d-1)} dy. \tag{38}$$

If z_n denotes the expected value of X_n then, by using equations 37 and setting $c_0 = 1$, one obtains the following recurrence relations for z_n,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k z_{n-k} = \triangle^n(z_0) = z_0^{n+1}(-1)^n \prod_{k=0}^{n} c_k \tag{39}$$

and the representation

$$z_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k z_0^{k+1} \prod_{i=0}^{k} c_i. \tag{40}$$

For $|x| \leq 2r$, the symmetric difference $B_r \triangle (x + B_r)$ is given by

$$\psi_r(|x|) = 2\left(m(B_r) - r^{d-1}\omega_d \int_0^{\arccos(|x|/(2r))} \sin^d(t)\, dt \right). \tag{41}$$

Equation 41 yields $\mathbb{E}_{\mathbb{P}_L}(\psi(X_n)) \sim \psi'(0) z_n = \mathrm{Per}(B_r) z_n$ as $n \to \infty$. In particular, if f is radially decreasing around x_0 with support in B_L, then $\mathbb{E}_{\mathbb{P}_L}(\|f^{\sigma_1 \cdots \sigma_n} - f^*\|_{L^1}) \sim z_n \|\mathrm{Per}(f^* > t)\|_{L^1}$ as $n \to \infty$. Setting $d = 1$, we obtain the exact representation

$$\begin{aligned} z_n &= 4L \sum_{k=0}^{n} \binom{n}{k} (-1)^k \int_0^{z_0/4L} t^k\, dt & (42) \\ &= 4L \int_0^{z_0/4L} (1-t)^n\, dt & (43) \\ &= \frac{4L}{n+1} \left[1 - (1 - z_0/4L)^{n+1} \right] & (44) \end{aligned}$$

and the exact asymptotic

$$\mathbb{E}_{\mathbb{P}_L}(\|f^{\sigma_1 \cdots \sigma_n} - f^*\|_{L^1}) \sim \frac{4L}{n} \|\mathrm{Per}(f^* > t)\|_{L^1}. \tag{45}$$

Setting $d = 1$ in Proposition 2, we see that the rate of convergence estimate is sharp for $d = 1$. We can strengthen this result by analyzing the characteristic function of the random variables X_n. We have

$$\begin{aligned} \mathbb{E}_{\mathbb{P}_L}\left(e^{itX_n}\right) &= 1 + 4L \int_0^{z_0/4L} (1-x)^n \sum_{j=1}^{\infty} \frac{j(4Lx)^{j-1}(it)^j}{j!} dx & (46) \\ &= 1 + it \int_0^{z_0} e^{xit}(1 - x/4L)^n dx & (47) \end{aligned}$$

and

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}_L}\left(e^{itnX_n}\right) = 1 + it\int_0^\infty e^{xit}e^{-x/4L}dx. \tag{48}$$

This shows that nX_n converges weakly to an exponential distribution with mean $4L$. Unfortunately, the constants c_n are not easily computable for dimensions $d > 1$, but the method of proof for $d = 1$ can be used to make a connection between this problem and the Hausdorff moment problem. Setting $z_0 = 1$ in equation 40 gives

$$(-1)^n \triangle^n \prod_{i=0}^{j} c_i = \mathbb{E}_{\mathbb{P}_L}(X_n^j)\prod_{i=0}^{j} c_i \geq 0. \tag{49}$$

Hausdorff has shown that if 49 holds then there exists a unique probability measure μ such that

$$\int_0^1 x^k d\mu(x) = \prod_{i=0}^{k} c_i. \tag{50}$$

This gives

$$\mathbb{E}_{\mathbb{P}_L}(X_n^k) = \prod_{i=0}^{k-1}(c_i/z_0)^{-1}\int_0^1 x^{k-1}(1-xz_0)^n d\mu(x) \tag{51}$$

and

$$\mathbb{E}_{\mathbb{P}_L}(e^{itX_n}) = 1 + it\int_0^1 (1-xz_0)^n\psi(itxz_0)d\mu(x) \tag{52}$$

with

$$\psi(x) = \sum_{i=0}^{\infty}\frac{x^k}{(k+1)!}\prod_{i=0}^{k} c_i^{-1}. \tag{53}$$

We have

$$\liminf_{n\to\infty} \mathbb{E}_{\mathbb{P}_L}((nX_n)^k) \geq \liminf_{n\to\infty} n\mu(x < n^{-1})e^{-z_0}\prod_{i=0}^{k-1}(c_i/z_0)^{-1} \tag{54}$$

for all $k \geq 1$ but equation 37 gives $\mathbb{E}_{\mathbb{P}_L}(X_n - X_{n+1}) \geq c_1\mathbb{E}_{\mathbb{P}_L}(X_n)$ which implies $\mathbb{E}_{\mathbb{P}_L}(X_n) \leq c_1^{-1}n^{-1}$; consequently, if $\liminf_{n\to\infty} n\mu(x < n^{-1}) > 0$ then we have $z_n \sim cn^{-1}$ for some numerical constant c depending on d.

References

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